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Vertex-removal, vertex-addition and different notions of similarity for vertices of a graph. (English) Zbl 1512.05281

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The authors compare different notions of similarity of vertices of a graph, where the similarity is defined in terms of either vertex removal or vertex addition. Specifically, for a graph $G = (V, E)$ and $u, v \in V$, u is X-similar to v if the property $P(X)$ holds, for the following notions of X and $P(X)$:

- automorphism: $\exists\ \sigma \in \mathrm{Aut}(G)$ such that $\sigma(u) = v$;
- removal: $G - u \cong G - v$;
- attachment: $G^u\mathbf{1} \cong G^v\mathbf{1}$, where $G^x\mathbf{1}$ is the graph obtained from G by adding an additional vertex, and an edge from this vertex to $x \in V$;
- NDS: $\mathrm{NDS}(u, G) = \mathrm{NDS}(v, G)$, where $\mathrm{NDS}(x, G)$ is the neighborhood degree sequence of x;
- attachment index: $\varrho(G^u\mathbf{1}) = \varrho(G^v\mathbf{1})$, where $\varrho(G)$ is the spectral radius of G;
- removal index: $\varrho(G - u) = \varrho(G - v)$;
- cospectral: $G - u$ is cospectral to $G - v$;
- attachment CS: $\Pi(G^u\mathbf{1}) = \Pi(G^v\mathbf{1})$, where $\Pi(G)$ is the complementarity spectrum of G;
- removal CS: $\Pi(G - u) = \Pi(G - v)$.

The authors prove theorems that place these variations of removal and attachment similarity into a hierarchy, with automorphism similarity (equivalent to attachment similarity) being the strongest notion. Where possible, they give examples of graphs having a small number of vertices that satisfy a weaker notion of similarity but not a stronger one: for instance, they state that removal similarity and cospectral similarity coincide on graphs of order at most 7, and give an example of a tadpole graph on 8 vertices having a pair of cospectrally similar vertices that are not removal similar.

Since testing for graph isomorphism is in general a difficult problem, the ability to test for the existence of non-similar vertices for some weaker notion than automorphism similarity is useful. In particular, CS-similarity is offered as one more tool for this purpose in this paper, since it is stronger than index-similarity, yet not as computationally expensive as testing for graph isomorphism; specifically, an example of a graph on 12 vertices is discussed, where neither NDS-similarity nor cospectral similarity help in determining whether two attachment index-similar vertices are automorphically similar, but CS-similarity is found helpful.

It is a natural question to ask whether a weaker notion of similarity implies a stronger notion under additional restrictions on the vertices. The authors show that this is indeed possible in some cases. For instance, attachment similarity is in general stronger than attachment index similarity. But, it is proved that for a starlike tree, two leaves are attachment similar if and only if they are attachment index-similar.

There are also cases where the (non-)existence of the reverse implication is unclear: while attachment similarity implies attachment CS-similarity, the authors state that they are not aware of the existence of any connected graph for which attachment similarity is different from attachment CS-similarity. Similar remarks are made for removal similarity vs. removal CS-similarity.

Two additional remarks:

- On page 5187, read "NDS-similar" in place of "NSD-similar".
- For Theorem 2.1, the proof that (ii) implies (iv) may require further elaboration. Alternatively, one can show that (ii) implies (i), as below. Here, (i) is the statement that u is automorphically similar to v, (ii) is the statement that u is attachment similar to v, and (iv) is the statement that $G^uH \cong G^vH$ for all rooted graphs H, where G^xH is the graph obtained by adding an edge joining the vertex x of G to the root of H.

  First, we check that if u and v are attachment similar, then they have the same degree. Let $d_u =$

$\deg_G(u)$ and $d_v = \deg_G(v)$. Since $G^u\mathbf{1} \cong G^v\mathbf{1}$ by hypothesis, the sets $S_u = \{x \in V(G^u\mathbf{1}) : \deg_{G^u\mathbf{1}}(x) = d_u + 1\}$ and $S_v = \{y \in V(G^v\mathbf{1}) : \deg_{G^v\mathbf{1}}(y) = d_u + 1\}$ have the same cardinality. Note that $S_u = \{u\} \cup T$, where $T = \{x \in V(G) : \deg_G(x) = d_u + 1\}$. If $v \in T$, then $S_v = T \setminus \{v\}$, and if $v \notin S_v \cup T$, then $S_v = T$. Either case contradicts that $|S_u| = |S_v|$. Hence, $S_v = \{v\} \cup T$, so $d_u + 1 = d_v + 1$. Thus, $d_u = d_v$.

Next, let $\phi\colon G^u\mathbf{1} \to G^v\mathbf{1}$ be an isomorphism. Let $x_0 = u$, and define $x_i = \phi(x_{i-1})$ for $i \geq 1$. If $x_1 = v$, then we are done, so suppose that $x_1 \neq v$. Then, $x_1 \in T$. Note that $\phi(T) \subseteq S_v \setminus \{x_1\}$. In particular, there exists $i > 1$ such that $\phi(x_i) = v$, by the pigeonhole principle. Then, iterating the map ϕ appropriately, we obtain an isomorphism $\psi\colon G^u\mathbf{1} \to G^v\mathbf{1}$ such that $\psi(u) = v$.

This proves that (ii) implies (i). Since it is easy to see that (i) implies (iv), this is an alternative way to complete the required chain of implications to prove Theorem 2.1.

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MSC:

05C50 Graphs and linear algebra (matrices, eigenvalues, etc.)
15A42 Inequalities involving eigenvalues and eigenvectors

Cited in **2** Documents

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References:

[1] Harary, F.; Palmer, E., On similar points of a graph, J Math Mech, 15, 623-630 (1966) · Zbl 0141.21404

[2] Lauri, J.; Scapellato, R., Topics in graph automorphisms and reconstruction (2016), Cambridge: Cambridge University Press, Cambridge · Zbl 1347.05001

[3] Kocay, W., Attaching graphs to pseudo-similar vertices, J Austral Math Soc Ser A, 36, 53-58 (1984) · Zbl 0533.05032

[4] Bondy, JA; Murty, USR., Graph theory (2008), Springer Verlag · Zbl 1134.05001

[5] Seeger, A.; Sossa, D., Extremal problems involving the two largest complementarity eigenvalues of a graph, Graphs Comb, 36, 1-25 (2020) · Zbl 1434.05095

[6] Bell, FK; Rowlinson, P., The change in index of a graph resulting from the attachment of a pendant edge, Proc R Soc Edinburgh Sec A, 108, 67-74 (1988) · Zbl 0719.05047

[7] Dalfó, C.; Fiol, MA; Garriga, E., A differential approach for bounding the index of graphs under perturbations, Electron J Comb, 18, 1, 172 (2011) · Zbl 1235.05086

[8] Rowlinson, P., The spectrum of a graph modified by the addition of a vertex, Univ Beograd Publ Elektrotehn Fak Ser Mat, 3, 67-70 (1992) · Zbl 0776.05074

[9] Oliveira, ER; Stevanovic, D.; Trevisan, V., Spectral radius ordering of starlike trees, Linear Multilinear Algebra, 68, 991-1000 (2020) · Zbl 1455.05047

[10] Seeger, A., Complementarity spectral analysis of connected graphs, Linear Algebra Appl, 543, 205-225 (2018) · Zbl 1387.05160

[11] Schwenk, A. Almost all trees are cospectral. New Directions in the Theory of Graphs. New York: Academic Press; 1973, p. 275-307. · Zbl 0261.05102

[12] Herndon, WC; Ellzey, ML., Isospectral graphs and molecules, Tetrahedron, 31, 99-107 (1975)

[13] Collatz, L.; Sinogowitz, U., Spektren endlicher Grafen, Abh Math Sem Univ Hamburg, 21, 63-77 (1957) · Zbl 0077.36704

[14] Dalfó, C.; van Dam, ER; Fiol, MA., On perturbations of almost distance-regular graphs, Linear Algebra Appl, 435, 2626-2638 (2011) · Zbl 1222.05152

[15] Cvetković, D.; Rowlinson, P.; Simić, S., Eigenspaces of graphs (1997), Cambridge: Cambridge University Press, Cambridge · Zbl 0878.05057

[16] Godsil, CD, Smith, J. Strongly cospectral vertices. September 2017, arXiv:1709.07975 [math.CO].

[17] Godsil, CD; McKay, BD., Feasibility conditions for the existence of walk-regular graphs, Linear Algebra Appl, 30, 51-61 (1980) · Zbl 0452.05045

[18] Seeger, A., Eigenvalue analysis of equilibrium processes defined by linear complementarity conditions, Linear Algebra Appl, 292, 1-14 (1999) · Zbl 1016.90067

[19] Fernandes, R.; Judice, J.; Trevisan, V., Complementarity eigenvalue of graphs, Linear Algebra Appl, 527, 216-231 (2017) · Zbl 1365.05171

[20] Seeger, A.; Sossa, D., On cardinality of complementarity spectra of connected graphs, Linear Algebra Appl, 614, 5-23 (2021)

· Zbl 1459.05186

[21] Pinto da Costa, A.; Seeger, A., Cone-constrained eigenvalue problems: theory and algorithms, Comput Optim Appl, 45, 25-57 (2010) · Zbl 1193.65039

[22] Kirkpatrick, DG; Klawe, MM; Corneil, DG., Pseudosimilarity in trees, J Comb Theory Ser B, 34, 323-339 (1983) · Zbl 0536.05056

[23] Borgatti, SP, Grosser, TJ. Structural equivalence: meaning and measures. International Encyclopedia of the Social and Behavioral Sciences. 2nd ed. 2015, p. 621-625.